Solution 5

1. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

   Solution. $\mathbb{Z}$ is a closed subset so it is complete. On the other hand, the closure of $\mathbb{Q}$ is $\mathbb{R}$, it is not complete.

2. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting

$$d(n,m) = \left| \frac{1}{n} - \frac{1}{m} \right| .$$

   (a) Show that it is not a complete metric.

   (b) Describe how to make it complete by adding one new point.

   Solution. The sequence $\{n\}$ is a Cauchy sequence in this metric but it has no limit. Its completion is obtained by adding an ideal point called ∞ and define $\tilde{d}(x,y) = d(x,y)$ when $x, y \in \mathbb{N}$ and $\tilde{d}(x,\infty) = 1/x$ for all $x \in \mathbb{N}$ and $\tilde{d}(\infty,\infty) = 0$.

3. Optional. Let (X,d) be a metric space. Fixing a point $p \in X$, for each x define a function

$$f_x(z) = d(z,x) - d(z,p).$$

   (a) Show that each f_x is a bounded, uniformly continuous function in X.

   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X,d) to $C_b(X)$ (shorthand for $C_b(X,\mathbb{R})$) . In other words,

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

   (c) Deduce from (b) the completion theorem.

   This approach is much shorter than the proof given in notes. However, it is not so inspiring.

   Solution.

   (a) From $|f_x(z)| = |d(z,x) - d(z,p)| \le d(x,p)$, and from $|f_x(z) - f_x(z')| \le |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)| \le 2d(z,z')$, it follows that each f_x is a bounded, uniformly continuous function in X.

   (b) $|f_x(z) - f_y(z)| = |d(z,x) - d(z,y)| \le d(x,y)$, and equality holds taking $z = x$. Hence

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

   (c) Let $Y_0 = \{f_x : x \in X\} \subset C_b(X)$. Let Y be the closure of Y_0 in the complete metric space $(C_b(X), \rho)$ with sup-norm ρ. Then (Y,ρ) is a completion of (X,d).

4. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

   Solution. Since T^k is a contraction, there is a unique fixed point $x \in X$ such that $T^k x = x$. Then $T^{k+1}x = T^k T x = Tx$ shows that Tx is also a fixed point of T^k. From the uniqueness of fixed point we conclude $Tx = x$, that is, x is a fixed point for T. Uniqueness is clear since any fixed point of T is also a fixed point of T^k.

5. Show that the equation $x = \frac{1}{2}\cos^2 x$ has a unique solution in $\mathbb{R}$.

   Solution. Let $Tx = \frac{1}{2}\cos^2 x$. Then $T'(x) = -\frac{1}{2}\sin 2x$ so $|T'| \le 1/2$. It follows that $|Tx - Ty| \le \frac{1}{2}|x-y|$, T is a contraction. By the fixed point theorem, we conclude that $x = \frac{1}{2}\cos^2 x$ has a unique solution.

6. Show that the equation $2x\sin x - x^4 + x = 0.001$ has a root near $x = 0$.

   Solution. Here $\Psi(x) = 2x\sin x - x^4$. We need to find some r, γ so it is a contraction. We have

$$
\begin{aligned}
|\Psi(x_1) - \Psi(x_2)| &= \left|2x_1(\sin x_1 - \sin x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^4 - x_2^4)\right| \\
&= \left|2x_1 \cos c(x_1 - x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^2 + x_2^2)(x_1 + x_2)(x_1 - x_2)\right| \\
&\le (2r + r + (2r^2)(2r))|x_1 - x_2| \ .
\end{aligned}
$$

   Taking $r = 1/4, \gamma = 2r + r + (2r^2)(2r) = 13/16 < 1$. By the Perturbation of Identity Theorem, the equation $2x\sin x - x^4 + x = y$ is solvable for any y satisfying $|y| \le R = (1-\gamma)r = 0.0468$, including $y = 0.001$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \ne 0$. Show that there exists some $\rho > 0$ such that

$$
Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),
$$

   is a contraction. This provides a justification for Newton's method in finding roots for an equation.

   Solution. $T'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$. Since f is C^2 and $f(x_0) = 0, f'(x_0) \ne 0$, it follows that T is C^1 in a neighborhood of x_0 with $T(x_0) = x_0, T'(x_0) = 0$ and there exists some $\rho > 0$

$$
|T'(x)| \le \frac{1}{2}, \quad x \in [x_0 - \rho, x_0 + \rho].
$$

   As a result, T is a contraction in $[x_0 - \rho, x_0 + \rho]$. By Contraction Mapping Principle, there is a fixed point for T. From the definition of T, this fixed point is a root for the equation $f(x) = 0$.